THE HOMOLOGY GROUPS OF SOME TWO-STEP NILPOTENT LIE ALGEBRAS ASSOCIATED TO SYMPLECTIC VECTOR SPACES

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Let $\Phi : V \mapsto \Phi(V)$ be a polynomial functor from the category of vector spaces (over a field $\mathbb{F}$ of characteristic zero) to the category of Lie algebras. In this paper, we study the functors $H_k(\Phi) : V \mapsto H_k(\Phi(V))$ from vector spaces to vector spaces obtained by composing Φ with the kth Lie algebra homology group functor. These functors are also polynomial functors, and are best studied by expressing them as explicit Schur functors.

The simplest example is obtained by taking Φ to be the identity functor, which assigns to a vector space V the Lie algebra V with vanishing bracket. In this case, $H_k(\Phi)$ is the kth exterior power.

A more complicated example was investigated by Sigg [10]. Take Φ to be the free 2-step nilpotent Lie algebra functor $V \mapsto \mathsf{Lie}_2(V) = V \oplus \Lambda^2 V$, with bracket

$$[(v_1, a_1), (v_2, a_2)] = (0, v_1 \wedge v_2), \qquad v_i \in V, a_i \in \Lambda^2 V.$$

If λ is a Young diagram, let S^λ be the Schur functor associated to λ (cf. Fulton-Harris [1]); in particular, $\mathsf{S}^{(1^k)}$ is the kth exterior power, while $\mathsf{S}^{(k)}$ is the kth symmetric power. Let λ^* be the conjugate partition of λ, defined by $\lambda^*_i = \sup\{j \mid \lambda_j \geq i\}$. Introduce the set $\mathcal{O}_k$ of Young diagrams such that λ is self-conjugate, $\lambda = \lambda^*$, and $2k = |\lambda| + \sup\{i \mid \lambda_i \geq i\}$. Sigg proves that

$$H_k(\mathsf{Lie}_2) \cong \sum_{\lambda \in \mathcal{O}_k} \mathsf{S}^\lambda .$$

For example, $H_1(\mathsf{Lie}_2)$ is the identity functor, $H_2(\mathsf{Lie}_2) \cong \mathsf{S}^{(2,1)}$, and $H_3(\mathsf{Lie}_2) \cong \mathsf{S}^{(3,1^2)} \oplus \mathsf{S}^{(2^2)}$.

In this paper, we prove an analogue of Sigg's result. Let H be the symplectic vector space $\mathbb{F} \oplus \mathbb{F}$, with symplectic form $\langle (a,b), (c,d) \rangle = ad - bc$. Let $\mathsf{L}_\mathsf{H}(V)$ be the Lie algebra $(\mathsf{H} \otimes V) \oplus \mathsf{S}^2(V)$, with bracket

$$[(v_1, w_1; a_1), (v_2, w_2; a_2)] = (0, 0; v_1 \cdot w_2 - v_2 \cdot w_1), \qquad (v_i, w_i) \in \mathsf{H} \otimes V, a_i \in \mathsf{S}^2(V).$$

The homology $H_k(\mathsf{L}_\mathsf{H})$ is more complicated than that of Lie_2, even when H is two-dimensional, and we have not been able to calculate it completely. As an illustration,

$$H_k(\mathsf{L}_\mathsf{H}) \cong \begin{cases} \big(\mathsf{H}_0 \otimes \mathsf{S}^{(0)}\big), & k = 0, \\ \big(\mathsf{H}_1 \otimes \mathsf{S}^{(1)}\big), & k = 1, \\ \big(\mathsf{H}_2 \otimes \mathsf{S}^{(1^2)}\big) \oplus \big(\mathsf{H}_1 \otimes \mathsf{S}^{(3)}\big), & k = 2, \\ \big(\mathsf{H}_3 \otimes \mathsf{S}^{(1^3)}\big) \oplus \big(\mathsf{H}_2 \otimes \mathsf{S}^{(3,1)}\big) \oplus \big(\mathsf{H}_0 \otimes \mathsf{S}^{(4)}\big) & k = 3, \end{cases}$$

where H_k is the kth symmetric power of H (and is thus $k+1$-dimensional).

The Lie algebras $\mathsf{L}_\mathsf{H}(V)$ satisfy Poincaré duality, since their associated simply connected Lie group is contractible and contains a cocompact lattice. (We owe this remark to P. Etingof.)

For example, $\mathsf{S}^\lambda(\mathbb{F})$ is nonzero only if λ has length 1; we see that

$$H_\bullet(\mathsf{L}_\mathsf{H}(\mathbb{F})) \cong \mathbb{F} \oplus \mathsf{H}[1] \oplus \mathsf{H}[3] \oplus \mathbb{F}[4],$$

where $V[k]$ is the vector space V shifted into degree k. In higher dimensions, Poincaré duality is difficult to see directly.

As we explain in [4], the homology groups $H_k(\mathsf{L}_\mathsf{H})$ are closely related to the E_2-terms of the Leray-Serre spectral sequence for the fibrations $\mathcal{M}_{g,n} \to \mathcal{M}_g$ (or, in genus 1, of the fibrations $\mathcal{M}_{1,n} \to \mathcal{M}_{1,1}$). In particular, the summand $\mathsf{H}_0 \otimes \mathsf{S}^{(4)}$ of $H_3(\mathsf{L}_\mathsf{H})$ gives rise to the relation in $H^4(\overline{\mathcal{M}}_{1,4}, \mathbb{Q})$ discovered in [3].

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1. Lie $\mathbb{S}$-algebras

In this section, we recall parts of the formalism of operads, referring to Getzler-Jones [5] for further details. This formalism is closely related to Joyal's theory of species and analytic functors (Joyal [7]).

1.1. Definition. An $\mathbb{S}$-module is a functor from $\mathbb{S}$, the groupoid formed by taking the union of the symmetric groups $\mathbb{S}_n$, $n \geq 0$, to the category of vector spaces.

Associated to an $\mathbb{S}$-module A is the functor from the category of vector spaces to itself,

$$V \longmapsto \mathsf{A}(V) = \sum_{k=0}^{\infty} \big(\mathsf{A}(k) \otimes V^{\otimes k}\big)_{\mathbb{S}_k}.$$

This is a generalization of the notion of a Schur functor, which is the special case where A is an irreducible representation of $\mathbb{S}_n$.

1.2. Definition. A polynomial functor Φ is a functor from the category of vector spaces to itself such that the map $\Phi : \mathrm{Hom}(V, W) \to \mathrm{Hom}(\Phi(V), \Phi(W))$ is polynomial for all vector spaces V and W. An analytic functor Φ is a direct image of polynomial maps.

To an analytic functor Φ, we may associate the $\mathbb{S}$-module

$$\mathsf{A}(n) = \Phi(\mathbb{F}x_1 \oplus \cdots \oplus \mathbb{F}x_n)_{(1,\dots,1)} \subset \Phi(\mathbb{F}x_1 \oplus \cdots \oplus \mathbb{F}x_n),$$

the summand of $\Phi(\mathbb{F}x_1 \oplus \cdots \oplus \mathbb{F}x_n)$ homogeneous of degree 1 in each of the generators x_i. We call A the $\mathbb{S}$-module of Taylor coefficients of Φ. The following theorem is proved in Appendix A of Macdonald [9].

1.3. Theorem. *There is an equivalence of categories between the category of $\mathbb{S}$-modules and the category of analytic functors: to an $\mathbb{S}$-module, we associate the functor $V \mapsto \mathsf{A}(V)$, while to an analytic functor Φ, we associate its $\mathbb{S}$-module of Taylor coefficients.*

Any $\mathbb{S}$-module A extends to a functor on the category of finite sets and bijections: if S is a finite set of cardinality n, we have

$$\mathsf{A}(S) = \Bigg(\sum_{\substack{f:[n]\to S \\ \text{bijective}}} \mathsf{A}(n) \Bigg)_{\mathbb{S}_n},$$

where $[n] = \{1, \ldots, n\}$. The category of $\mathbb{S}$-modules has a monoidal structure, defined by the formula

$$(\mathsf{A} \circ \mathsf{B})(n) = \sum_{k=0}^{\infty} \Big(\mathsf{A}(k) \otimes \sum_{f:[n]\to[k]} \mathsf{B}(f^{-1}(1)) \otimes \ldots \otimes \mathsf{B}(f^{-1}(k)) \Big)_{\mathbb{S}_k}.$$

This definition is motivated by the composition formula $(\mathsf{A} \circ \mathsf{B})(V) \cong \mathsf{A}(\mathsf{B}(V))$.

1.4. Definition. An **operad** is a monoid in the category of $\mathbb{S}$-modules, with respect to the above monoidal structure.

We see that the structure of an operad on an $\mathbb{S}$-module A is the same as the structure of a triple on the associated analytic functor $V \mapsto \mathsf{A}(V)$.

The Lie operad Lie is the operad whose associated analytic functor is the functor taking a vector space to its free Lie algebra.

1.5. Definition. A Lie $\mathbb{S}$-algebra L is a left Lie-module in the category of $\mathbb{S}$-modules.

Lie $\mathbb{S}$-algebras are essentially the same things as analytic functors from the category of vector spaces to the category of Lie algebras; more precisely, they are the collections of Taylor coefficients of such functors.

If we unravel the definition of a Lie $\mathbb{S}$-algebra, we see that it is an $\mathbb{S}$-module L with $\mathbb{S}_n$-equivariant brackets

$$[-,-] : \operatorname{Ind}_{\mathbb{S}_k \times \mathbb{S}_{n-k}}^{\mathbb{S}_n} \mathsf{L}(k) \otimes \mathsf{L}(n-k) \longrightarrow \mathsf{L}(n)$$

for $0 \le k \le n$, such that if $a_i \in \mathsf{L}(n_i)$, $i = 1, 2, 3$, the following expressions vanish:

$$[a_1, a_2] - [a_2, a_1] \in \operatorname{Ind}_{\mathbb{S}_{n_1} \times \mathbb{S}_{n_2}}^{\mathbb{S}_n} \big(\mathsf{L}(n_1) \otimes \mathsf{L}(n_2) \big) \quad \text{and}$$

$$[a_1, [a_2, a_3]] + [a_2, [a_3, a_1]] + [a_3, [a_1, a_2]] \in \operatorname{Ind}_{\mathbb{S}_{n_1} \times \mathbb{S}_{n_2} \times \mathbb{S}_{n_3}}^{\mathbb{S}_n} \big(\mathsf{L}(n_1) \otimes \mathsf{L}(n_2) \otimes \mathsf{L}(n_3) \big).$$

If L is a Lie algebra, let $\mathsf{K}_\bullet(L)$ be the Chevalley-Eilenberg complex of L. Recall that $\mathsf{K}_k(L) = \Lambda^k L$ is the kth exterior power of L, and the differential $\partial : \mathsf{K}_k(L) \to \mathsf{K}_{k-1}(L)$ is given by the formula

$$\partial(a_1 \wedge \ldots \wedge a_k) = \sum_{1 \le i < j \le k} (-1)^{i-j+1} [a_i, a_j] \wedge a_1 \wedge \ldots \wedge \widehat{a_i} \wedge \ldots \widehat{a_j} \wedge \ldots \wedge a_k.$$

If L is a Lie $\mathbb{S}$-algebra, we obtain a sequence of analytic functors $V \mapsto \big(\mathsf{K}_\bullet(\mathsf{L}(V)), \partial\big)$. Define the Chevalley-Eilenberg complex of the Lie $\mathbb{S}$-algebra L to be the Taylor coefficients of this complex of analytic functors. In other words, $\mathsf{K}_k(\mathsf{L}) = \Lambda^k \circ \mathsf{L}$, where Λ^k is the $\mathbb{S}$-module

$$\Lambda^k(n) = \begin{cases} \mathsf{S}^{(1^k)}, & k = n, \\ 0, & k \neq n. \end{cases}$$

The differential $\partial : \mathsf{K}_k(\mathsf{L}(V)) \to \mathsf{K}_{k-1}(\mathsf{L}(V))$ is a natural transformation of analytic functors, and hence induces a map of $\mathbb{S}$-modules $\partial : \mathsf{K}_k(\mathsf{L}) \to \mathsf{K}_{k-1}(\mathsf{L})$. Clearly, we have $\partial^2 = 0$.

1.6. Definition. The kth homology group $H_k(\mathsf{L})$ of the Lie $\mathbb{S}$-algebra L is the kth homology group of the complex of $\mathbb{S}$-modules $(\mathsf{K}_\bullet(\mathsf{L}), \partial)$.

Thus, $H_k(\mathsf{L})$ is an $\mathbb{S}$-module for each $k \geq 0$.

2. Examples of Lie $\mathbb{S}$-algebras

As a left module over itself, the Lie operad Lie is a Lie $\mathbb{S}$-algebra; the corresponding analytic functor is the free Lie algebra functor. More generally, define Lie_d, $1 \le d \le \infty$, by

$$\mathsf{Lie}_d(n) = \begin{cases} \mathsf{Lie}(n), & n \le d, \\ 0, & n > d. \end{cases}$$

Each of these is a Lie $\mathbb{S}$-algebra; the brackets $\mathsf{Lie}_d(k) \otimes \mathsf{Lie}_d(n-k) \to \mathsf{Lie}_d(n)$ are defined as for Lie if $n \le d$, and of course vanish if $n > d$. The analytic functor associated to the Lie $\mathbb{S}$-module Lie_d is known as the free d-step nilpotent Lie algebra. We may view Sigg's theorem [10] as the calculation of the homology of the Lie $\mathbb{S}$-algebra Lie_2:

$$H_k(\mathsf{Lie}_2)(n) \cong \sum_{\{\lambda \in \mathcal{O}_k \mid |\lambda| = n\}} \mathsf{S}^\lambda .$$

Here, we use the same notation for the representation of the symmetric group $\mathbb{S}_n$ with the Young diagram λ as for the associated Schur functor S^λ.

The tensor product $R \otimes \mathsf{L}$ of a Lie $\mathbb{S}$-algebra L with a commutative algebra R is again a Lie $\mathbb{S}$-algebra. For example, let M be a differentiable manifold and let $\Omega^\bullet(M)$ be the differential graded algebra of complex differential forms. The homology of differential graded Lie $\mathbb{S}$-algebras is defined in a manner analogous to the definition of the homology of Lie $\mathbb{S}$-algebras, except that we must add to the Chevalley-Eilenberg differential ∂ the internal differential d in defining the homology groups. Let $\mathsf{F}(M,n)$ be the nth configuration space of M, defined by

$$\mathsf{F}(M,n) = \{ i : [n] \longrightarrow M \mid i \text{ is an embedding} \}.$$

Let $j(n) : \mathsf{F}(M,n) \to M^n$ be the open embedding of the configuration space. The resolution of the sheaf $j(n)_! j(n)^* \mathbb{C}$ on M^n constructed in [2] may be identified with the twist of the Chevalley-Eilenberg complex $\mathsf{K}_\bullet(\Omega^\bullet(M) \otimes \mathsf{Lie})(n)$ by the alternating character $\varepsilon(n)$ of $\mathbb{S}_n$. This yields natural isomorphisms

$$H^\bullet(\mathsf{F}(M,n),\mathbb{C})[n] \cong H_\bullet(\Omega^\bullet(M) \otimes \mathsf{Lie})(n) \otimes \varepsilon(n).$$

In particular, if M is a compact manifold whose cohomology over $\mathbb{C}$ is formal (such as a compact Kähler manifold), we see that

$$H^\bullet(\mathsf{F}(M,n),\mathbb{C})[n] \cong H_\bullet(H^\bullet(M,\mathbb{C}) \otimes \mathsf{Lie})(n) \otimes \varepsilon(n).$$

This reformulates a theorem of Totaro [11].

Another example of a Lie $\mathbb{S}$-algebra is associated to a symplectic vector space H with symplectic form $\langle -, - \rangle$: set $\mathsf{L}_{\mathsf{H}}(1) = \mathsf{H}$, and let $\mathsf{L}_{\mathsf{H}}(2)$ be the trivial representation $\mathsf{S}^{(2)}$ of $\mathbb{S}_2$. The Chevalley-Eilenberg complex of L_{H} is familiar from Weyl's construction of the irreducible representations of the symplectic group $\mathrm{Sp}(\mathsf{H})$: we have

$$\mathsf{K}_n(\mathsf{L}_{\mathsf{H}})(n+\ell) = \begin{cases} \operatorname{Ind}^{\mathbb{S}_{n+\ell}}_{\mathbb{S}_\ell \wr \mathbb{S}_2 \times \mathbb{S}_{n-\ell}} \Big(\big(\mathsf{S}^{(2)}\big)^{\otimes \ell} \otimes \mathsf{S}^{(1^{n-\ell})} \Big) \otimes \mathsf{H}^{\otimes(n-\ell)}, & \ell \ge 0, \\ 0, & \ell < 0. \end{cases}$$

In particular, $\mathsf{K}_n(\mathsf{L}_\mathsf{H})(n) \cong \mathsf{S}^{(1^n)} \otimes \mathsf{H}^{\otimes n}$, and

$$\mathsf{K}_n(\mathsf{L}_\mathsf{H})(n+1) \cong \sum_{1\le i<j\le n} \mathsf{S}^{(1^{n-1})} \otimes \mathsf{H}^{\otimes(n-1)} \otimes x_{ij}.$$

The differential $\partial : \mathsf{K}_n(\mathsf{L}_\mathsf{H})(n) \to \mathsf{K}_{n+1}(\mathsf{L}_\mathsf{H})(n)$ is given by

$$\partial(e_1 \otimes \ldots \otimes e_n) = \sum_{1\le i<j\le n} (-1)^{j-i+1} \langle e_i, e_j\rangle\, e_1 \otimes \ldots \otimes \widehat{e_i} \otimes \ldots \otimes \widehat{e_j} \otimes \ldots \otimes e_n \otimes x_{ij}.$$

If $\mathsf{S}^{\langle\lambda\rangle}(\mathsf{H})$ is the irreducible representation of $\mathrm{Sp}(\mathsf{H})$ associated to the Young diagram λ, it follows that

$$H_n(\mathsf{L}_\mathsf{H})(n) \cong \sum_{|\lambda|=n} \mathsf{S}^{\langle\lambda\rangle}(\mathsf{H}) \otimes \mathsf{S}^{\lambda^*}.$$

For example, if $\dim(\mathsf{H}) = 2$, denoting the kth symmetric power $\mathsf{S}^{\langle k\rangle}(\mathsf{H})$ of H by H_k, we have

$$\mathsf{K}_n(\mathsf{L}_\mathsf{H})(n) \cong \sum_{j=0}^{[\frac{n}{2}]} \mathsf{H}_{n-2j} \otimes \mathsf{S}^{(2^j,1^{n-2j})},$$

and $H_n(\mathsf{L}_\mathsf{H})(n) \cong \mathsf{H}_n \otimes \mathsf{S}^{(1^n)}$.

3. The Chevalley-Eilenberg complex of L_H

We now turn to the closer study of the Chevalley-Eilenberg complex of the Lie $\mathbb{S}$-algebra L_H. To this end, choose a basis $\{e_a \mid 1 \le a \le 2g\}$ for H, with symplectic form

$$\langle e_a, e_b\rangle = \eta_{ab}.$$

Let η^{ab} be the inverse matrix to η_{ab}:

$$\sum_{b=1}^{2g} \eta^{ab}\eta_{bc} = \delta^a_c.$$

Let V be a vector space with basis $\{E_i \mid 1 \le i \le r\}$; the symmetric square $\mathsf{S}^2(V)$ has basis $\{E_{ij} = E_iE_j \mid 1 \le i \le j \le r\}$.

The nilpotent Lie algebra $\mathsf{L}_\mathsf{H}(V) = (\mathsf{H} \otimes V) \oplus \mathsf{S}^2(V)$ has centre $\mathsf{S}^2(V)$, and the restriction of its Lie bracket to $\mathsf{H} \otimes V$ is

$$[e_a \otimes E_i, e_b \otimes E_j] = \eta_{ab}E_{ij}.$$

The Chevalley-Eilenberg complex of $\mathsf{L}_\mathsf{H}(V)$ is the graded vector space $\Lambda^\bullet(\mathsf{H}\otimes V)\otimes\Lambda^\bullet(\mathsf{S}^2(V))$. Denote by ε^a_i the operation of exterior multiplication by $e_a \otimes E_i$ on this complex, and let ι^i_a be its adjoint, characterized by the (graded) commutation relations

$$[\iota^i_a, \varepsilon^b_j] = \delta^i_j\delta^b_a.$$

Let $\varepsilon_{ij} = \varepsilon_{ji}$ be the operation of exterior multiplication by E_{ij} on the Chevalley-Eilenberg complex, and let ι^{ij} be its adjoint, characterized by the commutation relations

$$[\iota^{ij}, \varepsilon_{kl}] = \delta^i_k\delta^j_l + \delta^i_l\delta^j_k. \tag{3.1}$$

The differential ∂ of the Chevalley-Eilenberg complex and its adjoint ∂^* are given by the formulas

$$\partial = \tfrac{1}{2} \sum_{i,j,a,b} \eta^{ab} \varepsilon_{ij} \iota^i_a \iota^j_b, \quad \partial^* = -\tfrac{1}{2} \sum_{i,j,a,b} \eta_{ab} \varepsilon^a_i \varepsilon^b_j \iota^{ij}.$$

The following theorem is the most powerful idea in the calculation of the cohomology of nilpotent Lie algebras.

3.1. Theorem (Kostant [8])**.** *The kernel of the Laplacian $\Delta = [\partial^*, \partial]$ on the Chevalley-Eilenberg complex is isomorphic to the homology of the Lie algebra $\mathsf{L}_{\mathsf{H}}(V)$.*

Sigg [10] has calculated the Laplacian Δ for the free 2-step nilpotent Lie algebra $\mathsf{Lie}_2(V) = V \oplus \Lambda^2 V$. Our calculation is modelled on his, with some modifications brought on by the introduction of the symplectic vector space H.

The complexity of our notation is reduced by adopting the Einstein summation convention: indices $i, j, \dots$ lie in the set $\{1, \dots, r\}$, indices $a, b, \dots$ in the set $\{1, \dots, 2g\}$, and we sum over repeated pairs of indices if one is a subscript and one is a superscript.

3.2. Lemma. $\Delta = \varepsilon_{ij} \varepsilon^a_k \iota^i_a \iota^{jk} - \frac{1}{2} \eta_{ab} \eta^{cd} \varepsilon^a_i \varepsilon^b_j \iota^i_c \iota^j_d - g\, \varepsilon_{ij} \iota^{ij}$

Proof. We have

$$4[\partial^*, \partial] = -[\eta_{ab} \varepsilon^a_i \varepsilon^b_j \iota^{ij}, \eta^{cd} \varepsilon_{kl} \iota^k_c \iota^l_d] = -\eta_{ab} \eta^{cd} \varepsilon_{kl} [\varepsilon^a_i \varepsilon^b_j, \iota^k_c \iota^l_d] \iota^{ij} - \eta_{ab} \eta^{cd} \varepsilon^a_i \varepsilon^b_j [\iota^{ij}, \varepsilon_{kl}] \iota^k_c \iota^l_d.$$

The first term of the right-hand side is calculated as follows,

$$\begin{aligned} -\eta_{ab} \eta^{cd} [\varepsilon^a_i \varepsilon^b_j, \iota^k_c \iota^l_d] &= -\eta_{ab} \eta^{cd} \varepsilon^a_i [\varepsilon^b_j, \iota^k_c \iota^l_d] - \eta_{ab} \eta^{cd} [\varepsilon^a_i, \iota^k_c \iota^l_d] \varepsilon^b_j \\ &= \delta^k_j \varepsilon^a_i \iota^l_a + \delta^l_j \varepsilon^a_i \iota^k_a - \delta^k_i \iota^l_a \varepsilon^a_j - \delta^l_i \iota^k_a \varepsilon^a_j \\ &= \delta^k_j \varepsilon^a_i \iota^l_a + \delta^l_j \varepsilon^a_i \iota^k_a + \delta^k_i \varepsilon^a_j \iota^l_a + \delta^l_i \varepsilon^a_j \iota^k_a - 2g\, \delta^k_i \delta^l_j - 2g\, \delta^l_i \delta^k_j, \end{aligned}$$

while the second term is calculated by (3.1). □

4. The Casimir operator of GL(V)

If V is a vector space with basis $\{E_i \mid 1 \le i \le n\}$, the Lie algebra of $\mathrm{GL}(V)$ has basis $\{E^j_i \mid 1 \le i, j \le n\}$, with commutation relations

$$[E^j_i, E^l_k] = \delta^j_k E^l_i - \delta^l_i E^j_k.$$

The centre of $\mathrm{GL}(V)$ is spanned by $\mathcal{D} = E^i_i$, and the Casimir operator is the element of the centre of $U(\mathfrak{gl}(V))$ given by the formula

$$\Delta_{\mathrm{GL}(V)} = E^j_i E^i_j.$$

Let c_λ be the eigenvalue of the Casimir operator $\Delta_{\mathrm{GL}(V)}$ on the representation $\mathsf{S}^\lambda(V)$ of $\mathrm{GL}(V)$ with highest weight vector $\lambda = (\lambda_1, \dots, \lambda_r)$. Since the sum of the positive roots of $\mathrm{GL}(V)$ equals $2\rho = (2r-1, 2r-3, \dots, 3-2r, 1-2r)$, the theory of semisimple Lie algebras shows that, up to an overall factor,

$$c_\lambda = \|\lambda\|^2 + 2(\rho, \lambda) = \sum_{i=1}^r \lambda_i (\lambda_i + r - 2i + 1). \tag{4.2}$$

To see that this factor equals 1, observe that on the fundamental representation V, with highest weight $(1,0,\dots,0)$, the Casimir has eigenvalue r.

Given a Young diagram λ, let

$$n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i = \sum_{i\geq 1}\binom{\lambda_i^*}{2}.$$

4.1. Lemma. $c_\lambda = r|\lambda| + 2n(\lambda^*) - 2n(\lambda) = \sum_{i=1}^\infty \lambda_i^*(r-\lambda_i^*+2i-1)$

Proof. The proof follows from rearranging (4.2):

$$c_\lambda = r|\lambda| + 2\sum_{i=1}^r \binom{\lambda_i}{2} - 2\sum_{i=1}^r (i-1)\lambda_i. \quad \square$$

Recall the dominance order on Young diagrams:

$$\lambda \geq \mu \text{ if } |\lambda| = |\mu| \text{ and } \lambda_1+\cdots+\lambda_i \geq \mu_1+\cdots+\mu_i \text{ for all } i\geq 1.$$

If $\lambda\geq\mu$, then $\mu^*\geq\lambda^*$ (Macdonald, I.1.11 [9]).

4.2. Corollary. *If $\lambda\geq\mu$, then $c_\lambda\geq c_\mu$, with equality only if $\lambda=\mu$.*

Proof. If $\lambda\geq\mu$, we have

$$n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i = \sum_{i\geq 1}\sum_{j>i}\lambda_i = \sum_{i\geq 1}\Big(|\lambda| - \sum_{j=1}^i \lambda_i\Big) \leq \sum_{i\geq 1}\Big(|\mu| - \sum_{j=1}^i \mu_i\Big) = n(\mu).$$

Likewise, $n(\lambda^*)\geq n(\mu^*)$. In both cases, equality holds only if $\lambda=\mu$. The corollary now follows from Lemma 4.1. $\square$

4.3. Corollary. *On the tensor product $\mathsf{S}^\lambda(V)\otimes\mathsf{S}^\mu(V)$, the Casimir operator $\Delta_{\mathrm{GL}(V)}$ is bounded above by $c_\lambda+c_\mu+2(\lambda,\mu)$, with equality only on $\mathsf{S}^{\lambda+\mu}(V)\hookrightarrow\mathsf{S}^\lambda(V)\otimes\mathsf{S}^\mu(V)$.*

Proof. There can only be a nonzero morphism $\mathsf{S}^\nu(V)\hookrightarrow\mathsf{S}^\lambda(V)\otimes\mathsf{S}^\mu(V)$ if $\nu\leq\lambda+\mu$. It follows from Corollary 4.2 that

$$c_\nu \leq c_{\lambda+\mu} = \|\lambda+\mu\|^2 + 2(\rho,\lambda+\mu) = \|\lambda\|^2 + 2(\rho,\lambda) + \|\mu\|^2 + 2(\rho,\mu) + 2(\lambda,\mu). \quad \square$$

5. A formula for the Laplacian

In this section, we prove the following explicit formula for the Laplacian Δ on the Chevalley-Eilenberg complex $\mathsf{K}_\bullet(\mathsf{L}_\mathsf{H}(V))$.

5.1. Theorem. $\Delta = \frac{1}{2}\big(\Delta_{\mathrm{Sp}(\mathsf{H})} + \Delta_{\mathrm{GL}(V)} - (r+2g+1)\mathcal{D}\big)$

Theorem 5.1 will follow by combining the results of Lemmas 3.2, 5.2 and 5.3. The Lie algebra of $\mathrm{GL}(V)$ acts on $\mathsf{K}_\bullet(\mathsf{L}_\mathsf{H}(V))$ via the operations

$$E_i^j = \varepsilon_i^a\iota_a^j + \varepsilon_{ik}\iota^{jk}.$$

It follows that $\mathcal{D} = \varepsilon_i^a\iota_a^i + \varepsilon_{ij}\iota^{ij}$, while the Casimir operator for $\mathrm{GL}(V)$ acts on $\mathsf{K}_\bullet(\mathsf{L}_\mathsf{H}(V))$ as follows.

5.2. Lemma. $\Delta_{\mathrm{GL}(V)} = \varepsilon_i^a\varepsilon_j^b\iota_b^i\iota_a^j + 2\,\varepsilon_{ij}\varepsilon_k^a\iota_a^i\iota^{jk} + r\varepsilon_i^a\iota_a^i + (r+1)\varepsilon_{ij}\iota^{ij}$

Proof. We have

$$\begin{aligned} E_i^j E_j^i &= (\varepsilon_i^a \iota_a^j + \varepsilon_{ik}\iota^{jk})(\varepsilon_j^b \iota_b^i + \varepsilon_{jl}\iota^{il}) \\ &= \varepsilon_i^a \iota_a^j \varepsilon_j^b \iota_b^i + \varepsilon_i^a \iota_a^j \varepsilon_{jl}\iota^{il} + \varepsilon_{ik}\iota^{jk}\varepsilon_j^b \iota_b^i + \varepsilon_{ik}\iota^{jk}\varepsilon_{jl}\iota^{il} \\ &= -\varepsilon_i^a \varepsilon_j^b \iota_a^j \iota_b^i + r\varepsilon_i^a \iota_a^i + \varepsilon_{jl}\varepsilon_i^a \iota_a^j \iota^{il} + \varepsilon_{ik}\varepsilon_j^b \iota_b^i \iota^{jk} - \varepsilon_{ik}\varepsilon_{jl}\iota^{jk}\iota^{il} + (r+1)\varepsilon_{ij}\iota^{ij}. \end{aligned}$$

The (a)symmetries of $\varepsilon_{ik}\varepsilon_{jl}\iota^{jk}\iota^{il}$ force it to vanish, and the result follows. □

The Lie algebra of $\mathrm{GL}(\mathsf{H})$ acts on the Chevalley-Eilenberg complex of $\mathsf{L}_{\mathsf{H}}(V)$ by the operators

$$\{e_b^a = \varepsilon_i^a \iota_b^i \mid 1 \le a, b \le 2g\},$$

and the Lie subalgebra $\mathrm{Sp}(\mathsf{H}) \subset \mathrm{GL}(\mathsf{H})$ is spanned by the operators

$$\{e_{ab} + e_{ba} \mid 1 \le a \le b \le 2g\},$$

where $e_{ab} = \eta_{ac} e_b^c$. The Casimir operator of $\mathrm{Sp}(\mathsf{H})$ is given by the formula

$$\Delta_{\mathrm{Sp}(\mathsf{H})} = -\tfrac{1}{2}\eta^{ac}\eta^{bd}\big(e_{ab} + e_{ba}\big)\big(e_{cd} + e_{dc}\big) = -\eta^{ac}\eta^{bd}e_{ab}e_{cd} - \eta^{ac}\eta^{bd}e_{ab}e_{dc}.$$

5.3. Lemma. $\Delta_{\mathrm{Sp}(\mathsf{H})} = -\varepsilon_i^a \varepsilon_j^b \iota_b^i \iota_a^j - \eta_{ab}\eta^{cd}\varepsilon_i^a \varepsilon_j^b \iota_c^i \iota_d^j + (2g+1)\varepsilon_i^a \iota_a^i$

Proof. We have

$$\eta^{ac}\eta^{bd}e_{ab}e_{cd} = \eta^{ac}\eta^{bd}\eta_{aa'}\eta_{cc'}\varepsilon_i^{a'}\iota_b^i \varepsilon_j^{c'}\iota_d^j = -\eta_{ac}\eta^{bd}\varepsilon_i^a \iota_b^i \varepsilon_j^c \iota_d^j = \eta_{ac}\eta^{bd}\varepsilon_i^a \varepsilon_j^c \iota_b^i \iota_d^j - \varepsilon_i^a \iota_a^i$$

$$\eta^{ac}\eta^{bd}e_{ab}e_{dc} = \eta^{ac}\eta^{bd}\eta_{aa'}\eta_{dd'}\varepsilon_i^{a'}\iota_b^i \varepsilon_j^{d'}\iota_c^j = -\varepsilon_i^a \iota_b^i \varepsilon_j^b \iota_a^j = \varepsilon_i^a \varepsilon_j^b \iota_b^i \iota_a^j - 2g\varepsilon_i^a \iota_a^i. \quad \square$$

6. The case $g = 1$

In this section, we apply our results in the special case $g = 1$, in which the symplectic vector space H is two-dimensional. Recall Frobenius's notation for partitions: if $\alpha_1 > \cdots > \alpha_d \ge 0$ and $\beta_1 > \cdots > \beta_d \ge 0$,

$$(\alpha_1, \ldots, \alpha_d | \beta_1, \ldots, \beta_d)$$

is the partition of $\alpha_1 + \cdots + \alpha_d + \beta_1 + \cdots + \beta_d + d$ whose ith part equals $\alpha_i + i$ for $i \le d$, and $\sup\{j \mid \beta_j + j \ge i\}$ for $i > d$. For example, $(\alpha|\beta)$ corresponds to the hook $(\alpha + 1, 1^\beta)$, while $(d-1, d-2, \ldots, 1, 0 | d-1, d-2, \ldots, 1, 0)$ is the partition (d^d).

6.1. Definition. Let $\mathcal{P}_\ell$ be the set of partitions of 2ℓ of the form $(\alpha_1+1, \ldots, \alpha_d+1 | \alpha_1, \ldots, \alpha_d)$; thus $\alpha_1 + \cdots + \alpha_d + d = \ell$ and $\alpha_1 > \cdots > \alpha_d \ge 0$.

The following plethysm is Ex. I.5.10 of Macdonald [9]:

$$\mathsf{S}^{(1^\ell)} \circ \mathsf{S}^{(2)} = \sum_{\lambda \in \mathcal{P}_\ell} \mathsf{S}^\lambda . \tag{6.3}$$

6.2. Theorem. *The cohomology group $H_n(\mathsf{L}_{\mathsf{H}})(n+\ell)$ is zero except in the following cases:*

(i) *$\ell = 0$ and $n \ge 0$, in which case $H_n(\mathsf{L}_{\mathsf{H}})(n) \cong \mathsf{H}_n \otimes \mathsf{S}^{(1^n)}$;*

(ii) *$\ell > 0$ and $n \ge \ell + 2$.*

If $\ell > 0$ and $n \ge 2\ell + 2$, we have $H_n(\mathsf{L}_{\mathsf{H}})(n+\ell) \cong \displaystyle\sum_{\substack{\lambda \in \mathcal{P}_\ell \\ n \ge \ell + \alpha_1 + 1}} \mathsf{H}_{n-\ell} \otimes \mathsf{S}^{(1^{n-\ell})+\lambda}$.

Proof. The Chevalley-Eilenberg complex of $\mathsf{L}_\mathsf{H}(V)$ is bigraded, $\mathsf{K}_{k,\ell} = \Lambda^k(\mathsf{H}\otimes V)\otimes\Lambda^\ell\big(\mathsf{S}^2(V)\big)$, and since the differential ∂ is homogeneous of bidegree $(-2,1)$, the homology is also bigraded. In terms of this bigrading, we wish to calculate $H_{n-\ell,\ell}(\mathsf{L}_\mathsf{H})$; evidently, this vanishes unless $n \ge \ell$.

The plethysm (6.3) implies that

$$\mathsf{K}_{k,\ell}(\mathsf{L}_\mathsf{H})(n) = \sum_{j=0}^{[\frac{k}{2}]} \sum_{\lambda\in\mathcal{P}_\ell} \mathsf{H}_{k-2j} \otimes \mathsf{S}^{(2^j1^{k-2j})} \otimes \mathsf{S}^\lambda .$$

We will derive a lower bound for the Laplacian Δ on each summand.

Given a partition $\lambda\in\mathcal{P}_\ell$, we calculate that $c_\lambda = 2\ell r + 2\sum_{i=1}^d(\alpha_i+1) = 2(r+1)\ell$ and

$$(2^j1^{k-j},\lambda) \le \sum_{i=1}^j(\alpha_i+i+1) + 2\ell \le 3\ell + \tbinom{j+1}{2}.$$

On the summand $\mathsf{H}_{k-2j}\otimes\mathsf{S}^{(2^j1^{k-2j})}\otimes\mathsf{S}^\lambda$, we have $(r+3)\mathcal{D} = (r+3)(k+2\ell)$,

$$\begin{aligned}
\tfrac12\Delta_{\mathrm{GL}(V)} &\le \tfrac12 c_{(2^j,1^{k-2j})} + \tfrac12 c_\lambda + (2^j1^{k-2j},\lambda) \le \tfrac12 c_{(2^j,1^{k-2j})} + (r+3)\ell + \ell + \tbinom{j+1}{2}, \quad \text{and} \\
\Delta_{\mathrm{Sp}(\mathsf{H})} + c_{(2^j1^{k-2j})} &= \big\{(k-2j)^2 + 2(k-2j)\big\} + \big\{(k-j)(r-(k-j)+1) + j(r-j+3)\big\} \\
&= \tfrac12(r+3)k - j(k-j+1).
\end{aligned}$$

Combining all of these ingredients, we see that $\Delta \ge j\big(k-\frac32 j+\frac12\big)-\ell$. If $j>0$, the right-hand side is bounded below by $k-\ell-1$; unless $k\ge 2$ and $k\le \ell+1$, our summand does not contribute to $H_n(\mathsf{L}_\mathsf{H})(n+\ell)$. Equivalently, $n=k+\ell$ must lie in the interval $[\ell+2,2\ell+2]$.

It remains to consider the summands of $\mathsf{K}_{k,\ell}$ with $j=0$; these have the form

$$\mathsf{H}_k \otimes \sum_{\lambda\in\mathcal{P}_\ell} \mathsf{S}^{(1^k)}\otimes\mathsf{S}^\lambda .$$

On the summand $\mathsf{H}_k\otimes\mathsf{S}^{(1^k)+\lambda}$ of $\mathsf{H}_k\otimes\mathsf{S}^{(1^k)}\otimes\mathsf{S}^\lambda$, the operator $\Delta_{\mathrm{Sp}(\mathsf{H})}+\Delta_{\mathrm{GL}(V)}$ equals

$$\begin{aligned}
k(k+2) + c_{(1^k)} + c_\lambda + 2(1^k,\lambda) &= k(k+2) + k(r-k+1) + 2\ell(r+1) + 2\sum_{i=1}^k \lambda_i \\
&= (k+2\ell)(r+3) - \sum_{i=k+1}^{\alpha_1+1}\lambda_i,
\end{aligned}$$

while on all other irreducible components of $\mathsf{H}_k\otimes\mathsf{S}^{(1^k)}\otimes\mathsf{S}^\lambda$, it is strictly less. It follows that the Laplacian can only vanish on the summand $\mathsf{H}_k\otimes\mathsf{S}^{(1^k)+\lambda}$, and only at that when $k\ge\alpha_1+1$. $\square$

The following formula illustrates the behaviour of $H_n(\mathsf{L}_\mathsf{H})(n+\ell)$ when $n\in[\ell+2,2\ell+1]$

6.3. Proposition.

$$H_n(\mathsf{L}_\mathsf{H})(n+1) \cong \Big(\mathsf{H}_{n-1}\otimes\mathsf{S}^{(3,1^{n-2})}\Big) \oplus \begin{cases}\mathsf{H}_0\otimes\mathsf{S}^{(4)}, & n=3,\\ 0, & n\ne 3.\end{cases}$$

Proof. Pieri's formula shows that

$$
\begin{aligned}
\mathsf{K}_{n-1,1} &\cong \sum_{j=1}^{[\frac{n+1}{2}]} \mathsf{H}_{n-2j+1} \otimes \mathsf{S}^{(2^{j-1},1^{n-2j+1})} \otimes \mathsf{S}^{(2)} \\
&\cong \sum_{j=1}^{[\frac{n+1}{2}]} \mathsf{H}_{n-2j+1} \otimes \mathsf{S}^{(2^{j},1^{n-2j+1})} \oplus \sum_{j=1}^{[\frac{n}{2}]} \mathsf{H}_{n-2j+1} \otimes \mathsf{S}^{(3,2^{j-1},1^{n-2j})} \\
&\quad \oplus \sum_{j=2}^{[\frac{n+1}{2}]} \mathsf{H}_{n-2j+1} \otimes \mathsf{S}^{(3,2^{j-2},1^{n-2j+2})} \oplus \sum_{j=2}^{[\frac{n+1}{2}]} \mathsf{H}_{n-2j+1} \otimes \mathsf{S}^{(4,2^{j-2},1^{n-2j+1})} .
\end{aligned}
$$

On these four summands, the operator Δ equals $j(n-j+2)$, $j(n-j+3)-n-2$, $j(n-j+1)$ and $j(n-j+2)-n-3$, respectively. Thus, the only summands on which Δ vanishes are $\mathsf{H}_{n-1} \otimes \mathsf{S}^{(3,1^{n-2})}$, and $\mathsf{H}_0 \otimes \mathsf{S}^{(4)}$. □

The same method may be used in the case $\ell = 2$: we obtain

$$
H_n(\mathsf{L}_{\mathsf{H}})(n+2) \cong \Big(\mathsf{H}_{n-2} \otimes \mathsf{S}^{(4,2,1^{n-4})}\Big) \oplus \begin{cases} \mathsf{H}_1 \otimes \mathsf{S}^{(5,2)}, & n=5, \\ 0, & n \neq 5. \end{cases}
$$

Our search for a formula for $H_n(\mathsf{L}_{\mathsf{H}})(n+\ell)$ for all ℓ has been fruitless; nevertheless, it might be of interest to find one.

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